Chapter 2

THE FIBONACCI AND LUCAS NUMBERS

The great Italian mathematician, Leonardo of Pisa (c. 1170-1250), who is known today as Fibonacci (an abbreviation of filius Bonacci), expanded on the Arabic algebra of North Africa and introduced algebra into Europe. The solution of a problem in his book *Liber Abacci* uses the sequence

(F) $\qquad 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$

One of the many applications of this ***Fibonacci sequence*** is a theorem about the number of steps in an algorithm for finding the greatest common divisor of a pair of large integers. We study the sequence here because it provides a wonderful opportunity for discovering mathematical patterns.

The numbers shown in (F) are just the beginning of the unending Fibonacci sequence. The rule for obtaining more terms is as follows:

RECURSIVE PROPERTY. The sum of two consecutive terms in (F) is the term immediately after them.

For example, the term after 55 in (F) is 34 + 55 = 89 and the term after that is 55 + 89 = 144.

To aid in stating properties of the Fibonacci sequence, we use the customary notation F_0, F_1, F_2, F_3, ... for the integers of the Fibonacci sequence. That is, $F_0 = 0$, $F_1 = 1$, $F_2 = F_0 + F_1 = 1$, $F_3 = F_1 + F_2 = 2$, $F_4 = F_2 + F_3 = 3$, $F_5 = F_3 + F_4 = 5$, $F_6 = F_4 + F_5 = 8$, and so on. When F_n stands for some term of the sequence, the term just after F_n is represented by F_{n+1}, the term after F_{n+1} is F_{n+2}, and so on. Also, the term just before F_n is F_{n-1}, the term just before F_{n-1} is F_{n-2} and so on.

We now can define the Fibonacci numbers formally as the sequence F_0, F_1, ... having the two following properties.

INITIAL CONDITIONS $F_0 = 0$ and $F_1 = 1$.

RECURSION RULE $F_n + F_{n+1} = F_{n+2}$ for $n = 0, 1, 2, \ldots$.

Next let S_n stand for the sum of the Fibonacci numbers from F_0 through F_n. That is,

$$
\begin{aligned}
S_0 &= F_0 = 0 \\
S_1 &= F_0 + F_1 && = 0 + 1 && = 1 \\
S_2 &= F_0 + F_1 + F_2 && = 0 + 1 + 1 && = 2 \\
S_3 &= F_0 + F_1 + F_2 + F_3 && = S_2 + F_3 && = 2 + 2 = 4
\end{aligned}
$$

and in general, $S_n = F_0 + F_1 + F_2 + \ldots + F_n = S_{n-1} + F_n$.

We tabulate some of the values and look for a pattern.

n	0	1	2	3	4	5	6	7	...
F_n	0	1	1	2	3	5	8	13	...
S_n	0	1	2	4	7	12	20	33	...

Is there a relationship between the numbers on the third line of this table and the Fibonacci numbers? One pattern is that each of the terms of the sequence S_0, S_1, S_2, ... is 1 less than a Fibonacci number. Specifically, we have

$$S_0 = F_2 - 1 = 0$$
$$S_1 = F_3 - 1 = 1$$
$$S_2 = F_4 - 1 = 2$$
$$S_3 = F_5 - 1 = 4$$
$$S_4 = F_6 - 1 = 7$$
$$S_5 = F_7 - 1 = 12$$

and might conjecture that $S_n = F_{n+2} - 1$ for $n = 0, 1, 2, ...$ Does this formula hold for $n = 6$? Yes, since

$$\begin{aligned} S_6 &= F_0 + F_1 + F_2 + F_3 + F_4 + F_5 + F_6 \\ &= S_5 + F_6 \\ &= (F_7 - 1) + F_6 \\ &= (F_6 + F_7) - 1 \\ &= F_8 - 1. \end{aligned}$$

The first steps in proving our conjecture correct for all the terms in the unending sequence S_0, S_1, S_2, ... are rewriting the recursion formula $F_n + F_{n+1} = F_{n+2}$ as $F_n = F_{n+2} - F_{n+1}$ and then using this to replace each Fibonacci number in the sum S_n by a difference as follows:

$$S_n = F_0 + F_1 + F_2 + \ldots + F_{n-1} + F_n$$
$$S_n = (F_2 - F_1) + (F_3 - F_2) + (F_4 - F_3) + \ldots + (F_{n+1} - F_n) + (F_{n+2} - F_{n+1}).$$

Next we rearrange the terms and get

$$S_n = -F_1 + (F_2 - F_2) + (F_3 - F_3) + \ldots + (F_{n+1} - F_{n+1}) + F_{n+2}$$
$$S_n = -F_1 + 0 + 0 + \ldots + 0 + F_{n+2}$$
$$S_n = F_{n+2} - F_1 = F_{n+2} - 1.$$

Thus we have made our conjecture (that is, educated guess) into a theorem.

The fundamental relation $F_n = F_{n+2} - F_{n+1}$ can also be used to define F_n when n is a negative integer. Letting $n = -1$ in this formula gives us $F_{-1} = F_1 - F_0 = 1 - 0 = 1$. Similarly, one finds that $F_{-2} = F_0 - F_{-1} = 0 - 1 = -1$ and $F_{-3} = F_{-1} - \mathrm{F}_{-2} = 1 - (-1) = 2$. In this way one can obtain F_n for any negative integer n.

Some of the values of F_n for negative integers n are shown in the following table:

n	...	-6	-5	-4	-3	-2	-1
F_n	...	-8	5	-3	2	-1	1

Perhaps the greatest investigator of properties of the Fibonacci and related sequences was François Edouard Anatole Lucas (1842-1891). A sequence related to the F_n bears his name. The ***Lucas sequence***, 2, 1, 3, 4, 7, 11, 18, 29, 47, ..., is defined by

$$L_0 = 2,\ L_1 = 1,\ L_2 = L_1 + L_0,\ L_3 = L_2 + L_1,\ \ldots,\ L_{n+2} = L_{n+1} + L_n, \ldots .$$

Some of the many relations involving the F_n and the L_n are suggested in the problems below. These are only a very small fraction of the large number of known properties of the Fibonacci and Lucas numbers. In fact, there is a mathematical journal, *The Fibonacci Quarterly*, devoted to them and to related material.

Problems for Chapter 2

1. For the Fibonacci numbers F_n show that:

   (a) $F_3 = 2F_1 + F_0$. (b) $F_4 = 2F_2 + F_1$. (c) $F_5 = 2F_3 + F_2$.

2. The relation $F_{n+2} = F_{n+1} + F_n$ holds for all integers n and hence so does $F_{n+3} = F_{n+2} + F_{n+1}$. Combine these two formulas to find an expression for F_{n+3} in terms of F_{n+1} and F_n.

3. Find r, given that $F_r = 2F_{101} + F_{100}$.

4. Express $F_{157} + 2F_{158}$ in the form F_s.

5. Show the following:

   (a) $F_4 = 3F_1 + 2F_0$. (b) $F_5 = 3F_2 + 2F_1$.

6. Add corresponding sides of the formulas of the previous problem and use this to show that $F_6 = 3F_3 + 2F_2$.

7. Express F_{n+4} in terms of F_{n+1} and F_n.

8. Find s, given that $F_s = 3F_{200} + 2F_{199}$.

9. Find t, given that $F_t = 5F_{317} + 3F_{316}$.

10. Find numbers a and b such that $F_{n+6} = aF_{n+1} + bF_n$ for all integers n.

11. Show the following:

(a) $F_0 + F_2 + F_4 + F_6 = F_7 - 1$. (b) $F_0 + F_2 + F_4 + F_6 + F_8 = F_9 - 1$.
(c) $F_1 + F_3 + F_5 + F_7 = F_8$. (d) $F_1 + F_3 + F_5 + F_7 + F_9 = F_{10}$.

12. The relation $F_{n+2} = F_{n+1} + F_n$ can be rewritten as $F_{n+1} = F_{n+2} - F_n$. Use this form to find a compact expression for $F_a + F_{a+2} + F_{a+4} + F_{a+6} + \ldots + F_{a+2m}$.

13. Find p, given that $F_p = F_1 + F_3 + F_5 + F_7 + \ldots + F_{701}$.

14. Find u and v, given that $F_u - F_v = F_{200} + F_{202} + F_{204} + \ldots + F_{800}$.

15. Show the following:

(a) $F_4 = 3F_2 - F_0$. (b) $F_5 = 3F_3 - F_1$. (c) $F_6 = 3F_4 - F_2$.

16. Use the formulas $F_{n+4} = 3F_{n+1} + 2F_n$ and $F_{n+1} = F_{n+2} - F_n$ to express F_{n+4} in terms of F_{n+2} and F_n.

17. Show the following:

(a) $2(F_0 + F_3 + F_6 + F_9 + F_{12}) = F_{14} - 1$. (b) $2(F_0 + F_3 + F_6 + F_9 + F_{12} + F_{15}) = F_{17} - 1$.

18. Show the following:

(a) $2(F_1 + F_4 + F_7 + F_{10} + F_{13}) = F_{15}$. (b) $2(F_1 + F_4 + F_7 + F_{10} + F_{13} + F_{16}) = F_{18}$.

19. By addition of corresponding sides of formulas of the two previous problems, find expressions for:

(a) $2(F_2 + F_5 + F_8 + F_{11} + F_{14})$. (b) $2(F_2 + F_5 + F_8 + F_{11} + F_{14} + F_{17})$.

20. (i) Prove that $F_{n+3} - 4F_n - F_{n-3} = 0$ for $n \geq 3$.
(ii) Prove that $F_{n+4} - 7F_n + F_{n-4} = 0$ for $n \geq 4$.
*(iii) For $m \geq 4$, find a compact expression for $F_a + F_{a+4} + F_{a+8} + \ldots + F_{a+4m}$.

21. Evaluate each of the following sums:

(a) $\binom{2}{0} + \binom{1}{1}$. (b) $\binom{3}{0} + \binom{2}{1}$.

(c) $\binom{4}{0} + \binom{3}{1} + \binom{2}{2}$. (d) $\binom{5}{0} + \binom{4}{1} + \binom{3}{2}$.

(e) $\binom{6}{0} + \binom{5}{1} + \binom{4}{2} + \binom{3}{3}$. (f) $\binom{7}{0} + \binom{6}{1} + \binom{5}{2} + \binom{4}{3}$.

22. Find m, given that $\binom{9}{0} + \binom{8}{1} + \binom{7}{2} + \binom{6}{3} + \binom{5}{4} = F_m$.

23. Find r, s, and t given that:

(a) $\binom{2}{0}F_0 + \binom{2}{1}F_1 + \binom{2}{2}F_2 = F_r$. ($b$) $\binom{2}{0}F_1 + \binom{2}{1}F_2 + \binom{2}{2}F_3 = F_s$.

(c) $\binom{2}{0}F_7 + \binom{2}{1}F_8 + \binom{2}{2}F_9 = F_t$.

24. Find u, v, and w, given that:

(a) $\binom{3}{0}F_0 + \binom{3}{1}F_1 + \binom{3}{2}F_2 + \binom{3}{3}F_3 = F_u$.

(b) $\binom{3}{0}F_1 + \binom{3}{1}F_2 + \binom{3}{2}F_3 + \binom{3}{3}F_4 = F_v$.

(c) $\binom{3}{0}F_7 + \binom{3}{1}F_8 + \binom{3}{2}F_9 + \binom{3}{3}F_{10} = F_w$.

25. Find r,s, and t, given that:

(a) $(F_7)^2 + (F_8)^2 = F_r$. (b) $(F_8)^2 + (F_9)^2 = F_s$. (c) $(F_9)^2 + (F_{10})^2 = F_t$.

26. Find u, v, *and* w, given that:

(a) $(F_3)^2 - (F_2)^2 = F_u\ F_{u+3}$. ($b$) $(F_4)^2 - (F_3)^2 = F_v\ F_{v+3}$. ($c$) $(F_9)^2 - (F_8)^2 = F_w\ F_{w+3}$.

27. Let $L_0,\ L_1,\ L_2$,... be the Lucas sequence. Prove that

$$L_0 + L_1 + L_2 + L_3 + \ldots + L_n = L_{n+2} - 1.$$

28. Find r, given that $L_r = 2L_{100} + L_{99}$.

29. Find s, given that $L_s = 3L_{201} + 2L_{200}$.

30. Find t, given that $L_t = 8L_{999} + 5L_{998}$.

31. Show that $L_0 + L_2 + L_4 + L_6 + L_8 + L_{10} = L_{11} + 1$.

32. Find m, given that $L_0 + L_2 + L_4 + L_6 + \ldots + L_{400} = L_m + 1$.

33. Derive a formula for $L_1 + L_3 + L_5 + L_7 + \ldots + L_{2m+1}$.

34. Conjecture, and test in several cases, formulas for:

(a) $L_0 + L_3 + L_6 + L_9 + \ldots + L_{3m}$. (b) $L_1 + L_4 + L_7 + L_{10} + \ldots + L_{3m+1}$.

(c) $L_2 + L_5 + L_8 + L_{11} + \ldots + L_{3m+2}$. (d) $\binom{n}{0}L_k + \binom{n}{1}L_{k+1} + \binom{n}{2}L_{k+2} + \ldots + \binom{n}{n}L_{k+n}$.

35. Find r, s, and t, given that $F_2L_2 = F_r$, $F_3L_3 = F_s$, and $F_4L_4 = F_t$.

36. Find u, v, and w, given that $F_{10}/F_5 = L_u$, $F_{12}/F_6 = L_v$, and $F_{14}/F_7 = L_w$.

37. Evaluate the following:

(a) $(F_1)^2 - F_0F_2$. (b) $(F_2)^2 - F_1F_3$. (c) $(F_3)^2 - F_2F_4$. (d) $(F_4)^2 - F_3F_5$.

38. Evaluate the expressions of the previous problem with each Fibonacci number replaced by the corresponding Lucas number.

39. Which of the Fibonacci numbers F_{800}, F_{801}, F_{802}, F_{803}, F_{804}, and F_{805} are even?

40. Which of the Fibonacci numbers of the previous problem are exactly divisible by 3?